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13. ABSTRACT This paper discusses the application of the three and four parameter Gamma distributions to problems involving downtimes. A method for estimating parameter values from empirical data is given for the three parameter distribution. The procedure is illustrated by several examples and a computer program capable of performing the required calculations is included. In addition, this paper stresses the convenience of the Gamma distribution in downtime analysis. This distribution is especially useful in those cases in which convolutions are involved.			

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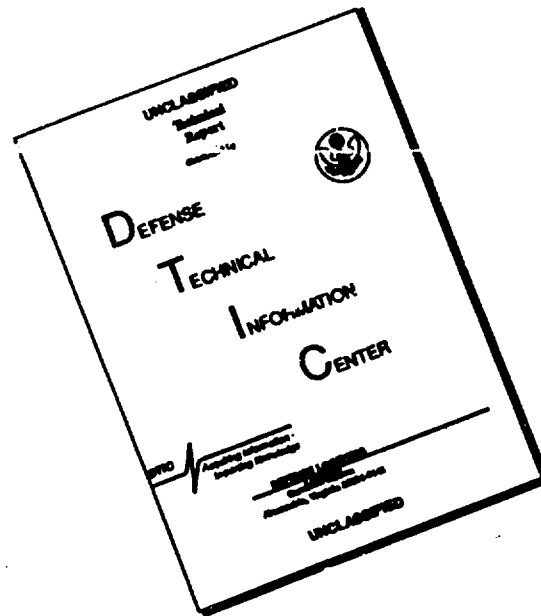
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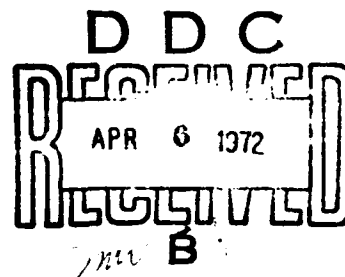
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RESEARCH REPORT

THE GAMMA DISTRIBUTION
IN DOWNTIME ANALYSIS

by

Stephen Michael Rolwing



Accomplished as Part of the
AMC Maintainability Engineering Graduate Program
USAMC Intern Training Center

and

Presented in Partial Fulfillment of the Requirements of the
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CHAPTER I

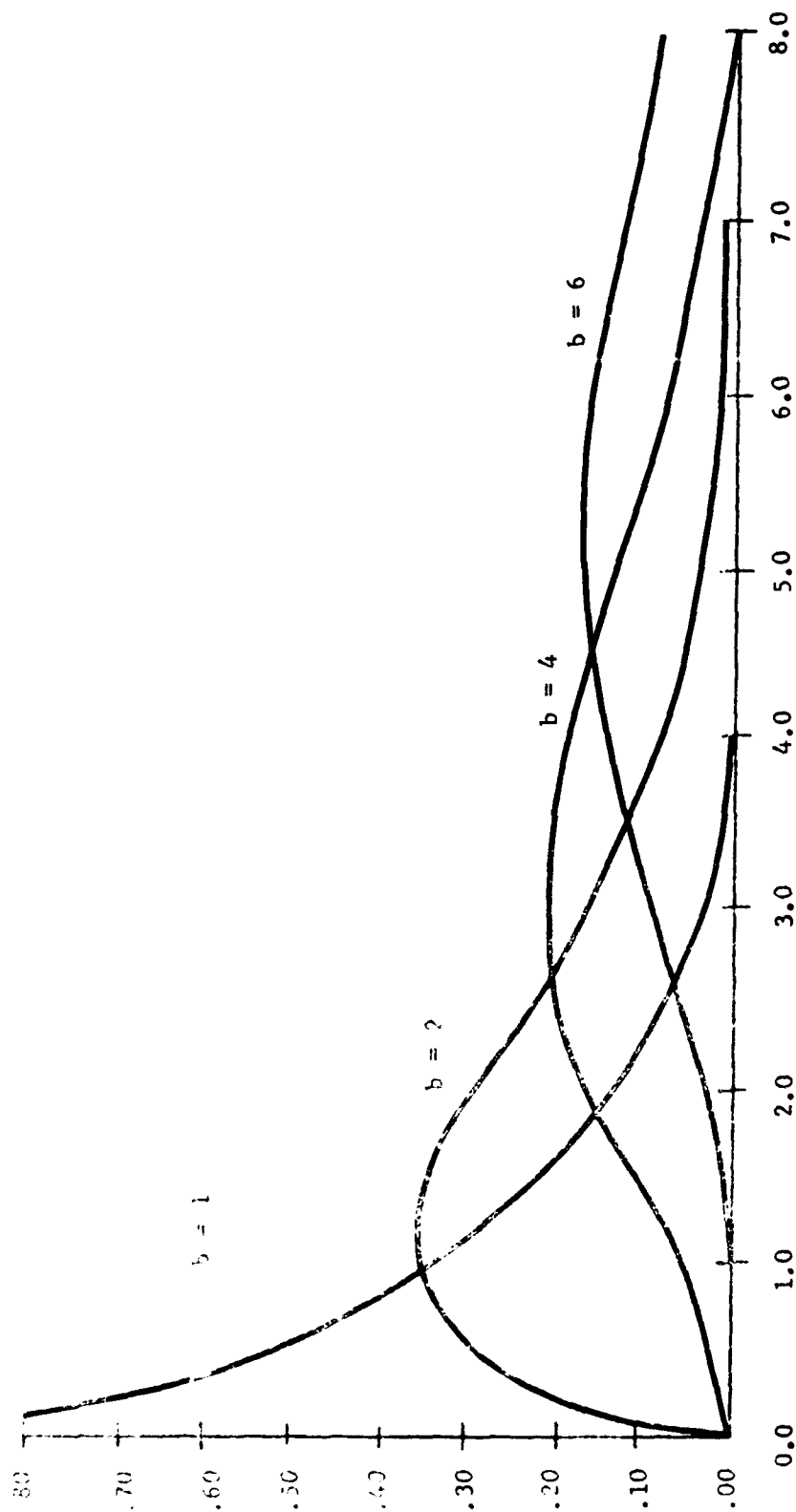
INTRODUCTION

Maintainability (M), as an engineering discipline, is progressing through its infancy and is now gathering sophistication and momentum. This growth can be seen in the increasing number of published articles introducing new methods, new applications of old methods, and new concepts slanted toward the field of maintainability. This paper can be placed in the second category. It is an old concept, applied in a different way to obtain the desired results.

The desired result in this case is the analysis of downtime data by fitting a distribution to the data. The old method is the familiar gamma distribution which has, in recent years, been the subject of lively conversations as a possible distribution of downtimes. It has been pointed out by many authors, Peterson (14)*, in particular, that the gamma distribution can assume a variety of different shapes. This is verified by Figure 1. In particular, note that when the shape parameter is equal to 1, the distribution is an exponential. Although the shape parameter is not restricted to an integer value, an Erlang distribution (1) results when it is. As the Erlang distribution has been used in queueing theory, some further attributes of the gamma distribution may be found there.

Peterson (14), in his paper, discusses the fitting of a gamma distribution to empirical data generated by a computer. Although he discusses the attributes of the gamma distribution in the analysis

* Numbers in parentheses refer to the material listed in the References at the end of this paper.



The Gamma Distribution for $a=1$, $b=1, 2, 4, 6$

FIGURE 1

of downtime to quite an extent, his method of parameter estimation leaves much to be desired statistically. This method consisted of plotting the empirical data and then "overlaying" this plot with plots of the gamma distribution with integer shape parameters. The gamma distribution which fitted best by this overlaying process was then used to solve for the scale parameter. The threshold parameter was assumed to be zero in all cases. Although the results were biased by this method, he does give an actual application of the gamma distribution fitted to downtime data.

The gamma distribution is attractive for two main reasons. The first is that the log-normal distribution, which is usually assumed to be the distribution of downtimes, is quite unwieldy and requires plotting on log-normal paper to ascertain that the data is actually log-normally distributed. The second reason is that many of the tools and methods being brought into use today sometimes require the manipulation of distributions in complex ways. For example, Pieruschka (15) develops the force of mortality of equipment with ideal repair and shows that a steady state value is obtained as the number of failures increase. He uses Laplace transforms to obtain the convolution of a gamma distribution with integer shape parameters and then integrates the resulting distribution to find the mean. These are the reasons why it is desirable to prove that the gamma distribution will represent the anticipated data, and still be mathematically simple.

Two downtime distributions can be spoken of in equipment analysis. The first is the distribution of elapsed time required to complete

a specific maintenance task. The second concerns the distribution of times required to complete different tasks within a given system, and this distribution is said to describe the system downtime. The random variable in the second case is, of course, composed of elements of both cases, but the randomness of the individual task times is usually neglected. This time element approach to the analysis of downtimes will not be used here; but, instead, we will only be interested in fitting a function to a number of data points.

The function to be fitted is the gamma distribution and parameter estimation is by maximum likelihood where certain terms of the maximum likelihood equations have been tabulated. These tables are entered with a parameter formed by the ratio of the arithmetic mean to the logarithmic mean and the shape parameter term is read out. Several more simple calculations will yield the threshold parameter and the scale parameter.

In obtaining probability levels, the gamma distribution has one of the disadvantages of the log-normal distribution in that tables must be consulted (8,13). All of the moments of the gamma distribution are functions of the three parameters, and possibly most important, a gamma distribution convoluted with itself, results in another gamma distribution.

To prove that the gamma distribution could be fitted to downtime data, a computer program was written to perform the estimation procedure. Random points from completely specified density functions as well as empirical data were fitted with a gamma distribution. When tested by the Kolmogorov-Smirnov goodness of fit test (11), there

was no significant difference between any of the sets of data and its corresponding gamma distribution approximation at the .05 level.

A brief literature survey will be given in Chapter II, with the ensuing chapter devoted to the development and use of the gamma distribution and the estimation procedure. The convolution of the gamma distribution as pertains to downtime data is developed, and several examples will be given in Chapter IV. Some possible extensions of and conclusions to the paper will then be discussed in Chapter V.

CHAPTER II

LITERATURE REVIEW

Much of the impetus for this paper resulted from Peterson's paper (14) which was a rough approximation of downtime data by the gamma distribution. Initially, work was started on the so-called generalized gamma distribution developed by Stacy (16) of the IBM Corporation and at times worked on by Stacy and Mihram (17), Harter (7), Harter and Moore (6), Parr and Webster (12), and Collins (3). The fourth parameter is a power of the exponential term in the three parameter gamma distribution. Although Stacy and Mihram and Collins have developed two different parameter estimation procedures, the methods are unwieldy and time consuming. The convolution of the generalized gamma distribution is given as a summation term. In attempting to fit the four parameter gamma distribution to data, it was soon realized that the advantage of a wider variety of possible shapes was not worth the additional complexity.

Although the method of moments in parameter estimation for the gamma distribution is quite simple, the greater confidence obtained by maximum likelihood is usually worthwhile. Several tables for solving the maximum likelihood equations are available. Among these are Chapman (2), Masuyama and Kuraiwa (1951), and Greenwood and Durand (5), the last being used in this paper. General discussions of the gamma distribution are contained in most statistics and reliability texts. Barlow (1) has developed the gamma distribution to quite an extent in reliability analysis, and develops the

mortality and hazard rates of the distribution. Probability tables for the Incomplete Gamma Function have been computed by Pearson (13) and Harter (8). Some of these discussions are presented in the next chapter in a general discussion of the gamma distribution.

CHAPTER III

THE GAMMA DISTRIBUTION

In this chapter, the gamma distribution will be investigated. The moments and the moment generating function along with the convolution of the gamma distribution will be given and the method of parameter estimation will be developed.

The gamma density function is given by

$$f(t) = \begin{cases} \frac{\left(\frac{t-c}{a}\right)^{b-1} e^{-\frac{t-c}{a}}}{a \Gamma(b)} & c < t < \infty \\ 0 & \text{otherwise} \end{cases}$$

where c is the threshold value,

a is the scale parameter,

b is the shape parameter, and

$\Gamma(n)$ is the gamma function with argument n (Table 2).

The cumulative distribution is

$$F(t_0) = \Pr(t < t_0) = I(u, p)$$

where $I(u, p)$ is the Incomplete Gamma Function Ratio and in Harter's notation (8)

$$p = b - 1$$

and

$$u = \frac{t_0 - c}{a^{b-1}}$$

The mean and variance are found to be

mortality and hazard rates of the distribution. Probability tables for the Incomplete Gamma Function have been computed by Pearson (13) and Harter (8). Some of these discussions are presented in the next chapter in a general discussion of the gamma distribution.

$$\mu = a b + c$$

and

$$\sigma^2 = a^2 b.$$

In general the k^{th} moment about the threshold parameter is

$$\mu_k = \frac{a^k \Gamma(b+k)}{\Gamma(b)} = a^k \sum_{j=0}^{k-1} (b+j).$$

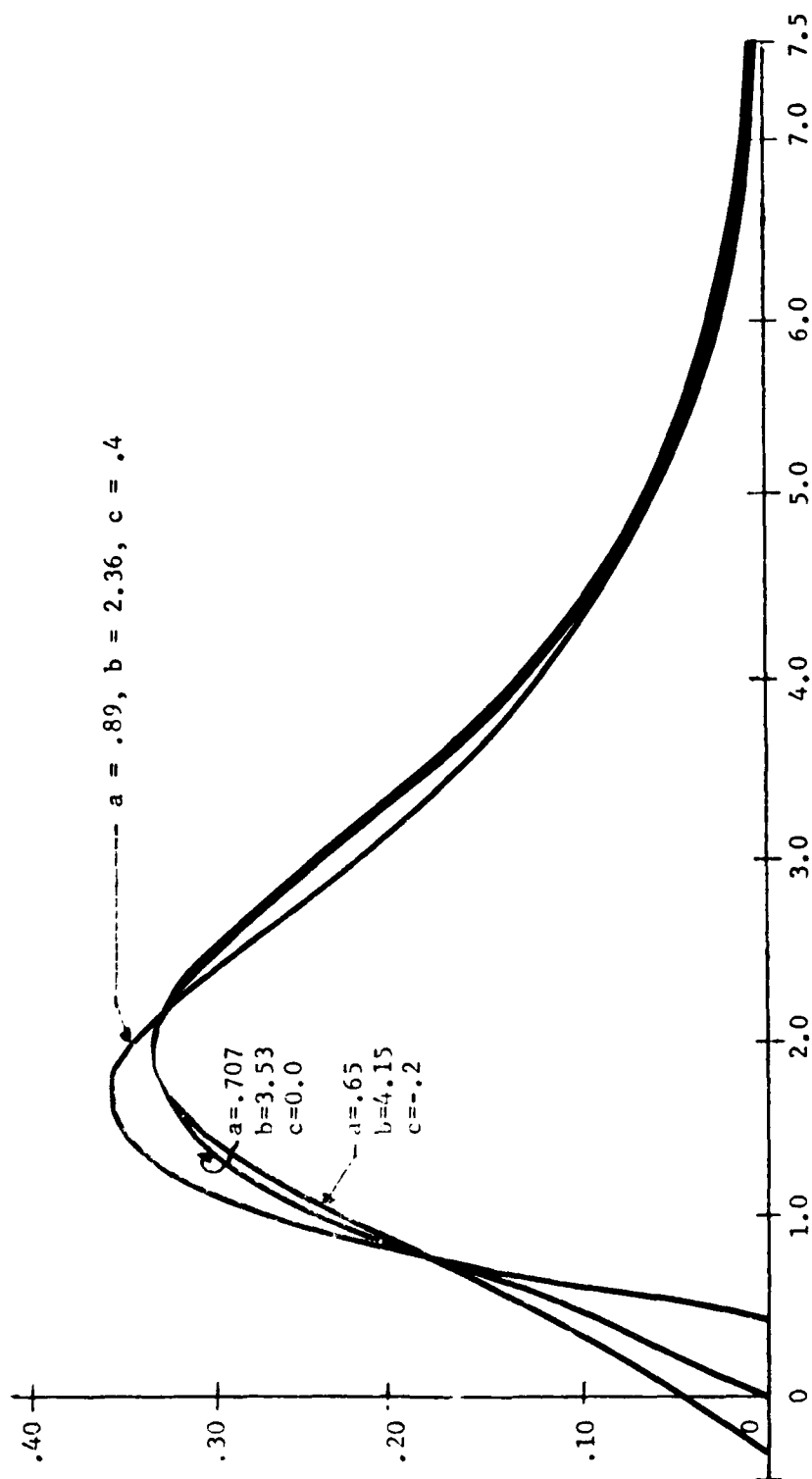
A point that will be used later should be noted here, and that is that a number of different combinations of scale parameter and shape parameter will yield approximately the same distribution. This can be seen from the equations for the mean and variance in which there are three unknowns. An illustration of this point is shown in Figure 2 where the respective parameters of the curves are

Curve #	c	a	b	Mean	Variance
1	-.2	.65	4.15	2.5	1.75
2	0	.707	3.53	2.5	1.76
3	.4	.89	2.36	2.5	1.87

Parameter Estimation by Maximum Likelihood

In the development of the maximum likelihood equations for parameter estimation (5), we form the joint probability of drawing n samples, $t_1, t_2, t_3, \dots, t_n$ from a gamma distribution with parameters c, a , and b .

$$L = \left(\frac{(t_1 - c)^{b-1} e^{-\frac{t_1 - c}{a}}}{a^b \Gamma(b)} \right) \left(\frac{(t_2 - c)^{b-1} e^{-\frac{t_2 - c}{a}}}{a^b \Gamma(b)} \right) \dots$$



Gamma Distributions

FIGURE 2

$$\left(\frac{(t_n - c)^{b-1} e^{-\frac{t_n - c}{a}}}{a^b \Gamma(b)} \right) = \frac{\prod_{i=1}^n (t_i - c)^{b-1} e^{-\sum_{i=1}^n \frac{t_i - c}{a}}}{a^{bn} [\Gamma(b)]^n}.$$

Taking the logarithm yields

$$\mathcal{L} = \ln L = (b-1) \sum_{i=1}^n \ln(t_i - c) - \sum_{i=1}^n \frac{t_i - c}{a} - bn \ln a - n \Gamma(b).$$

Differentiating with respect to the parameters to maximize the probability of obtaining the given sample results in

$$\frac{\partial \mathcal{L}}{\partial a} = \sum_{i=1}^n \frac{t_i - c}{a^2} - \frac{nb}{a} = 0$$

$$ab = \frac{\sum_{i=1}^n t_i - c}{n} \quad \text{Equation 1}$$

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_{i=1}^n \ln(t_i - c) - n \ln a - n \frac{\Gamma'(b)}{\Gamma(b)} = 0.$$

Substituting the value found for a in equation 1, one obtains

$$\ln b - \frac{\Gamma'(b)}{\Gamma(b)} = \ln \frac{\sum_{i=1}^n (t_i - c)}{n} - \frac{\sum_{i=1}^n \ln(t_i - c)}{n} \quad \text{Equation 2}$$

where $\Gamma'(b)$ is the first derivative of the gamma function with respect to b . Taking the partial with respect to the threshold parameter gives

$$\frac{\partial \mathcal{L}}{\partial c} = -(b-1) \sum_{i=1}^n \frac{1}{t_i - c} + n = 0$$

$$a(b-1) \sum_{i=1}^n \frac{1}{t_i - c} = n. \quad \text{Equation 3}$$

In equation 1 the product ab is seen to be the arithmetic mean

corrected for the threshold value. In equation 2, $[\Gamma'(b)] + [\Gamma(b)]$ is the tabulated digamma function (4), and equation 2 may be solved using a set of these tables and natural logarithms. However, the entire left-hand side of equation 2 has been tabulated in several different ways (2,5).

This then will be the method used to estimate the parameters. The right-hand side of equation 2 is simply the logarithm of the arithmetic mean minus the logarithm of the geometric mean. If we denote this right-hand side as Y and the left-hand side as $\eta(b)$, then

$$\eta(b) = Y.$$

Greenwood and Durand (5) have tabulated $b\eta(b) = b[\ln b - \Gamma'(b)/\Gamma(b)]$ versus $\eta(b) = Y$ to eight decimal places and their table is reproduced as Table 1.

The procedure will be to calculate the arithmetic and geometric means of the data and set Y equal to the difference of the logarithms of these two means. Enter Table 1 with this value and read $b\eta(b) = Yb$. Then solve for $b = Yb/Y$.

The scale parameter a is found from equation 1 and is equal to the arithmetic mean divided by b . Equation 3 indicates that some iteration process must be used to solve for the threshold value. Since this value must be found before the means can be calculated, a method of estimation which is free of terms containing a and b must be found. The following estimate was developed by fitting a straight line to the first three points on the curve generated by the expected ranks of the various ordered sample points. (Appendix 3 shows the development of this equation). The resulting expression is

$$c = \frac{Y_2 + Y_1}{2} - \frac{(Y_3 - Y_1)(Y_3 - Y_2)}{2(2Y_2 - Y_1 - Y_3)}$$

where Y_1 , Y_2 , and Y_3 are the first three ordered sample points.

Perhaps the most valuable asset of the gamma distribution is the ability to convolve one gamma distribution with another. The easiest proof of this is to derive the moment generating function and then find the distribution of the sum of random variables by multiplying their respective moment generating functions together. For the gamma distribution, the moment generating function is

$$\begin{aligned} E(e^{\theta t}) &= \int_{-\infty}^{\infty} \frac{t^{b-1} e^{-\frac{t}{a}} e^{\theta t}}{a^b \Gamma(b)} dt \\ &= \frac{1}{a^b \left(\frac{1}{a} - \theta\right)^b} \int_{-\infty}^{\infty} \frac{t^{b-1} e^{-\left(\frac{1}{a} - \theta\right)t}}{\Gamma(b)} dt \\ &= \frac{1}{(1 - a\theta)^b} (1) \left(\frac{1}{1 - a\theta}\right)^b \end{aligned}$$

Thus, a gamma distribution convolved with itself will result in a gamma distribution with threshold parameter $2c$, scale parameter a , and shape parameter $2b$. In general, when n gamma distributions are convolved together, if they have a common scale parameter, a gamma distribution results with threshold parameter $= \sum_{i=1}^n c_i$, scale parameter a , and shape parameter $= \sum_{i=1}^n b_i$. The restriction of equal scale parameters for the distributions being convolved will not be too great of an impairment in the treatment of downtimes since each set of data will have approximately the same range. Recalling that a set of data

can be represented by a variety of scale and shape parameter combinations, it is seen that sets of data with unequal scale parameters can be convolved by preassigning a common scale parameter to each. The shape parameter for each set of data would then be found by dividing the individual arithmetic means by the common scale parameter. This concept will be discussed in the following chapter in the form of an example.

CHAPTER IV

EXAMPLES

The first example in this chapter shows the steps necessary to fit a gamma distribution to a set of data. The second example will illustrate the method of convolving several sets of data with unequal scale parameters.

1. The following times were taken in a maintenance test on a particular system. We wish to perform an analysis on the data and fit a distribution to it so that we may speak of the probability of completing a future action or actions within a certain time.

1.7	4.6	2.3	3.0
4.0	3.4	1.3	3.7
3.3	2.9	3.8	2.2
2.6	2.0	2.7	4.3
3.5	2.5	3.1	

The first step in the analysis is to find the first three ordered points. They are $Y_1 = 1.3$, $Y_2 = 1.7$, $Y_3 = 2.0$, then

$$c = \frac{1.7 + 1.3}{2} - \frac{(2.0 - 1.3)(2.0 - 1.7)}{2[(2)(1.7) - 1.3 - 2.0]}$$

$$= 1.5 - \frac{.21}{.2} = 1.5 - 1.05 = .45$$

The arithmetic mean is found to be

$$\bar{x} = \frac{\sum t_i - c}{n} = \frac{48.35}{19} = 2.542.$$

the geometric mean is

$$\text{antiln} \left(\frac{\sum_{i=1}^n \ln(t_i - c)}{n} \right) = \text{antiln} \left(\frac{16.444}{19} \right) = 2.372,$$

$$\text{then } Y = \ln \left(\frac{2.542}{2.372} \right) = .0694$$

Entering Table 1 we find that for

$$Y = .0694, \quad Y_b = .5113.$$

Then

$$b = \frac{.5113}{.0694} = 7.37,$$

$$a = \frac{\bar{x}}{b} = \frac{2.54}{7.37} = .345,$$

$$\text{and } f(x) = \frac{(t - .45)^{6.37} e^{-\left(\frac{t - .45}{.345}\right)}}{.58}.$$

Entering the incomplete gamma function tables of Harter, where

$$p = b - 1 = 6.37 \approx 6.5$$

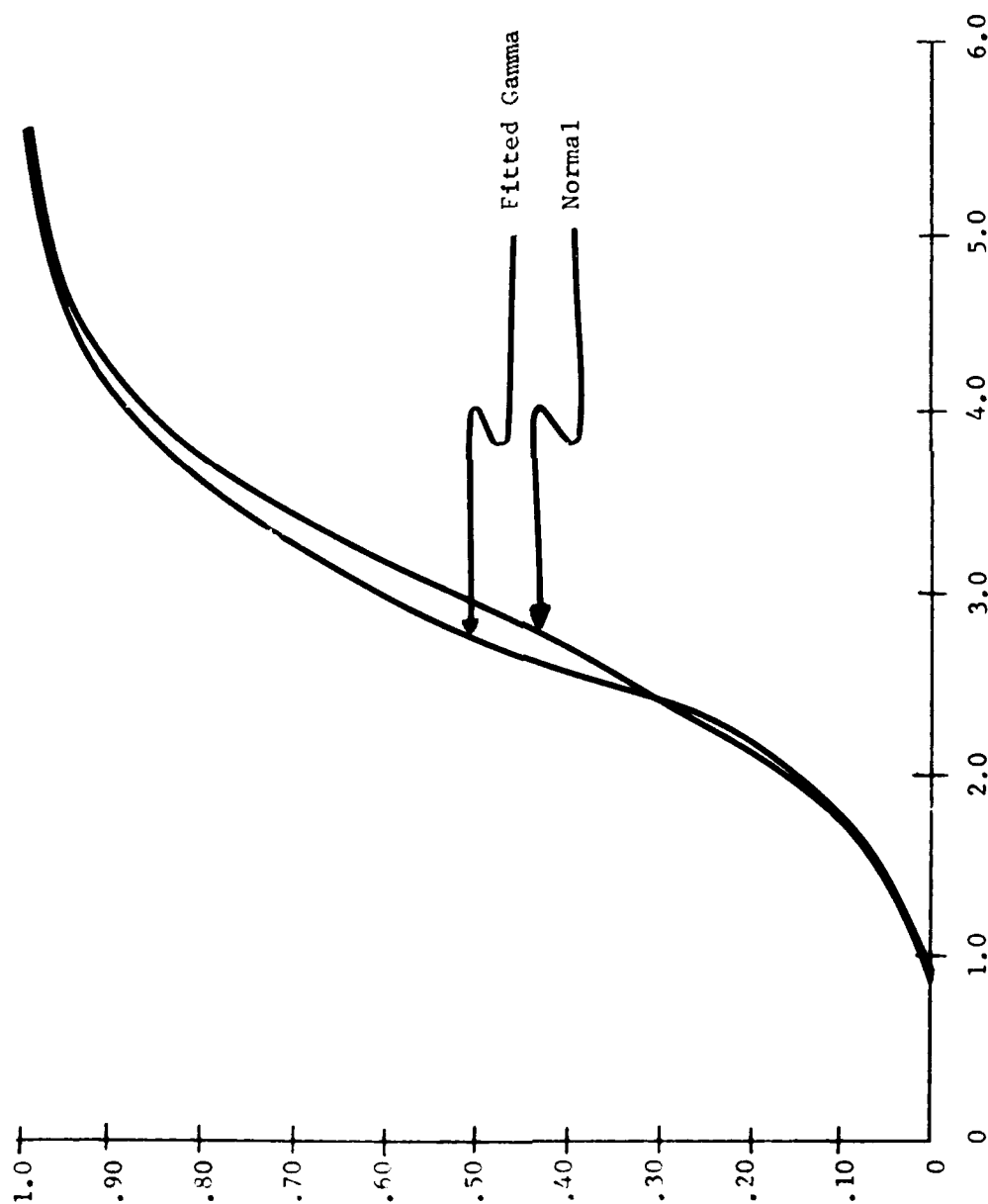
and

$$u = \frac{t - c}{a\sqrt{b}} = \frac{t - .45}{.935}$$

we can ascertain the cumulative probability for any given time and plot the graph of Figure 3. Also plotted on the graph is the curve from which the original data points were drawn, which in this case was a normal distribution with a mean of 3.0 and variance of 1.0. Performing the Kolmogorov-Smirnoff goodness of fit test (11)

$$D_{\text{MAX.}} = \text{MAX.} \left| F(t)_{\text{GAMMA}} - F(t)_{\text{NORMAL}} \right| = .06 < .301$$

where .301 is the critical value.



A GAMMA DISTRIBUTION FITTED TO NORMAL DATA

FIGURE 3

Therefore there is no significant difference at the .05 level of significance.

2. A second example will demonstrate the convolution of a number of sets of data. The data is as follows:

System 1:

3.13	1.96	3.84	2.33
2.48	3.00	1.36	3.38
4.64	4.04	2.62	1.72
2.87	2.16	3.67	4.28
3.52	3.25	2.75	

System 2:

4.57	4.04	5.76	6.87
5.59	3.81	4.71	5.43
3.54	5.00	5.29	4.24
6.19	6.46	3.13	5.96
4.85	5.15	4.41	

System 3:

3.11	1.42	1.67	2.22
2.45	2.11	.58	.89
1.89	1.78	2.58	2.90
3.42	2.73	2.00	1.55
1.27	2.33	1.10	

System 4:

7.61	5.16	4.39	5.52
5.35	6.00	7.50	6.48
4.50	7.06	6.16	5.68
6.65	5.84	8.07	6.32
5.93	4.74	6.84	

Fitting the gamma distribution to the four systems as in the first example, the various parameters are found to be,

System	c	a	b	Mean	Variance
1	.94	.46	4.5	3.0	.95
2	2.68	.54	4.3	5.0	1.26
3	2.03	.34	11.7	6.0	1.35
4	.19	.39	4.7	2.0	.715

$$\text{Setting } A = \frac{\sum_{i=1}^4 \Lambda_i}{4} = .43,$$

the b_i 's are then

$$b_1 = \frac{b_1}{A} = 4.80,$$

$$b_2 = \frac{b_2}{A} = 5.45,$$

$$b_3 = \frac{b_3}{A} = 9.24, \text{ and}$$

$$b_4 = \frac{b_4}{A} = 4.21.$$

Then the distribution of the convolved systems is a gamma distribution

with threshold parameter $C = \sum_{i=1}^4 c_i = 5.84$, scale parameter $A = .43$,

and shape parameter $B = \sum_{i=1}^4 b_i = 23.7$. This distribution has been

plotted in Figure 4 along with the convolution of the four data

generating distributions which were all normal with means of 3, 5, 6 and

2 and variances of 1, 1.3, 1.6 and 7.5.

Note in particular the parameters of the total distribution

Method	Mean	Variance	$t_{95\%}$	$t_{99\%}$
Convolved Gamma	16	4.4	19.74	21.59
Convolved Normal	16	4.65	19.56	21.03

The Kolmogorov-Smirnoff goodness of fit was again applied with $4 \times 19 = 76$ degrees of freedom. No significant difference could be found at the .05 level.

CHAPTER V

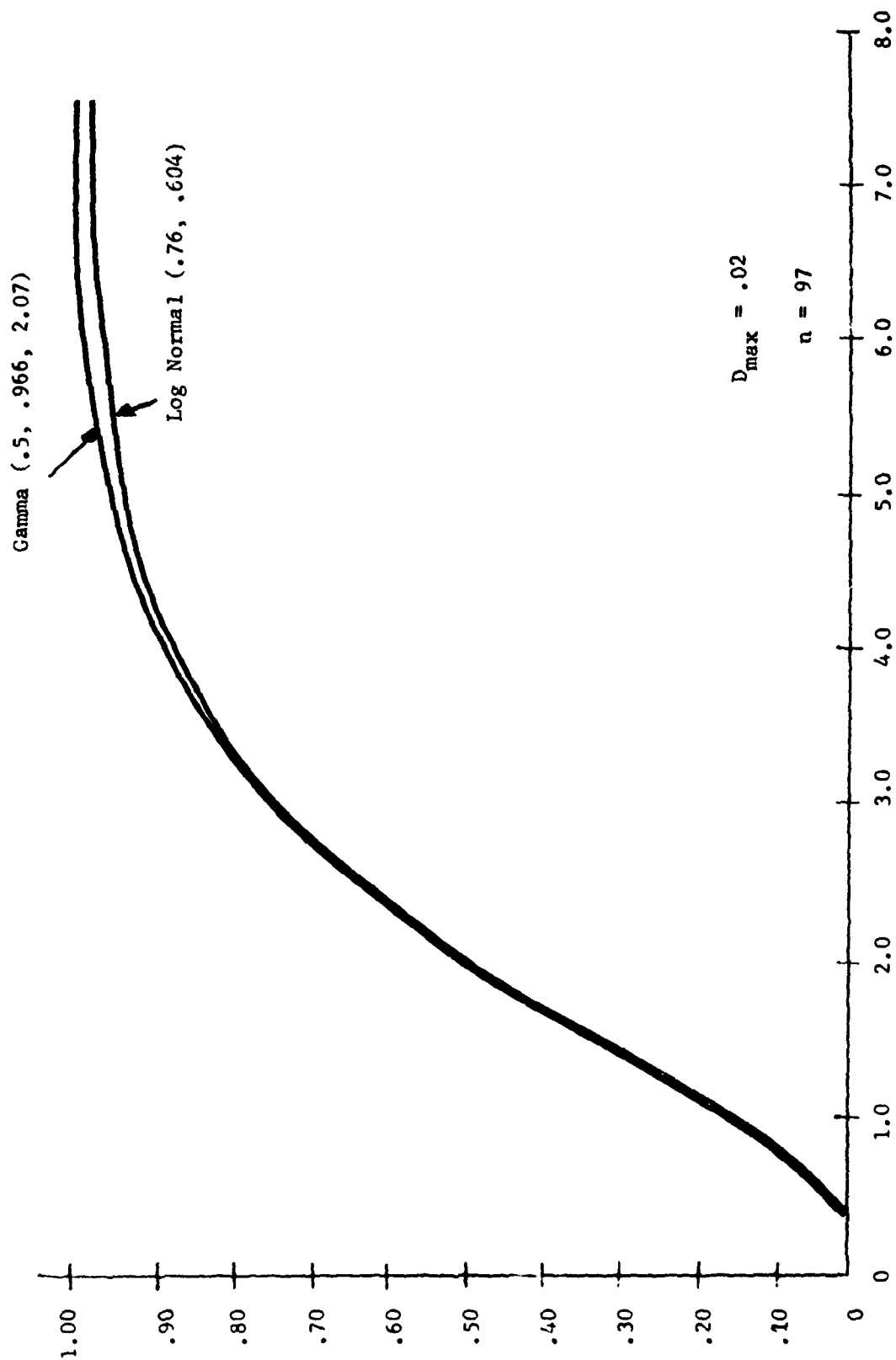
CONCLUSIONS

Although the gamma distribution approximations found by the author fitted extremely well, it is recommended that someone using the method go slowly and check the initial results before proceeding. This method will have important ramifications in the design of new systems since the designer will know beforehand that the downtimes will follow a gamma distribution of some sort. He can, therefore, feel free to use the models necessary for proper design of a logistical system without making a poorly based assumption as to the distribution of the downtimes. Also, with the many time critical repair action situations that occur in the space industry, a planning engineer can use the accurate probability models without employing an inaccurate repair time distribution.

The convolution technique developed for unequal scale parameters needs more validation than the single example presented. However, the potential of such a technique demands that it be investigated further.

For those who would still argue the validity of the log-normal distribution in maintainability work, Figure 4 has been provided. It is a gamma distribution fitted to a log-normal distribution with a mean of .76 and variance of .604.

In the estimation procedure, a and b were fixed by the data as soon as the threshold parameter was found, and varying c will vary the fit. The equation for c on page 13 is certainly not the best estimate. A possible alternative would be to plot the first few



A Gamma Distribution Fitted to Log-Normal Data

FIGURE 4

sample points versus their expected, or median, ranks and approximating the threshold parameter by eye. This would certainly lend self-assurance to the estimate.

Although the process can be hand operated, computer programs have been written for both the data fitting process, and for the convolution technique and are presented with the expected output in Appendices 1 and 2.

Greenwood and Durand (5) have also developed the formulas for the variances of a and b . Since we violated one of their original premises by estimating c from the data, these formulas will not be presented in order to avoid temptation of the reader.

Further areas of research in the line of using the gamma distribution for representing repair times would be to improve the estimate of the threshold parameter in order to obtain a better fit. Perhaps this should be developed for a better fit in the upper half or perhaps only the upper tail. The convolution technique has assumed equal weight for all of the systems, this could most certainly not be the case. Further research could be done on weighting the different sets of data by their frequency with respect to the overall system. Interesting possibilities also lie in the previous research done on the Erlang distribution (1) which is a special case of the gamma distribution. Perhaps some of the theories which have been developed in queueing theory about the Erlang distribution would be useful in maintenance queueing analysis.

It is hoped by the author that further research on the gamma distribution will be done and that the gamma distribution will be used more frequently in downtime analysis. Since the gamma distribution is quite

simple in form, especially when the shape parameter is an integer, it is easy to become familiar with the distribution and gain an insight into the data it represents. It is hoped that this paper will help make the gamma distribution an available tool of the Maintainability engineer.

TABLE 1

Y VERSUS Yb

Y	Yb						
.00	.50000000	.21	.53239205	.41	.55846165	.61	.58061573
.01	.50166108	.22	.53380204	.42	.55965491	.62	.58163629
.02	.50331088	.23	.53520025	.43	.56083849	.63	.58264927
.03	.50494925	.24	.53658674	.44	.56201253	.64	.58365479
.04	.50657603	.25	.53796161	.45	.56317712	.65	.58465291
.05	.50819111						
.06	.50979438	.26	.53932495	.46	.56433238	.66	.58564375
.07	.51138573	.27	.54067685	.47	.56547842	.67	.58662738
.08	.51296508	.28	.54201741	.48	.56661536	.68	.58760389
.09	.51453237	.29	.54334674	.49	.56774331	.69	.58857337
.10	.51608755	.30	.54466493	.50	.56886236	.70	.58953591
.11	.51763057	.31	.53597209	.51	.56997264	.71	.59049158
.12	.51916141	.32	.53726833	.52	.57107426	.72	.59144048
.13	.52068005	.33	.54855377	.53	.57216730	.73	.59238267
.14	.52218649	.34	.54982850	.54	.57325189	.74	.59331825
.15	.52368074	.35	.55109265	.55	.57432812	.75	.59424729
.16	.52516282	.36	.55234633	.56	.57539610	.76	.59516987
.17	.52663277	.37	.55358965	.57	.57645593	.77	.59608607
.18	.52809061	.38	.55482273	.58	.57750771	.78	.59699595
.19	.52953640	.39	.55604568	.59	.57855154	.79	.59789961
.20	.53097019	.40	.55725862	.60	.57958751	.80	.59879710

TABLE 1 - Continued

.81	.59958850	.95	.61237481	1.11	.62391639	1.26	.63448370
.82	.60057389	.97	.61317777	1.12	.62464951	1.27	.63515706
.83	.60145333	.98	.61397571	1.13	.62537836	1.28	.63582675
.84	.60232689	.99	.61476869	1.14	.62610298	1.29	.63649279
.85	.60319454	1.00	.61555677	1.15	.62682342	1.30	.63715523
.86	.60405665	1.01	.61633999	1.16	.62753971	1.31	.63781410
.87	.60491298	1.02	.61711841	1.17	.62825190	1.32	.63846943
.88	.60576369	1.03	.61789208	1.18	.62896003	1.33	.63912126
.89	.60660886	1.04	.61866105	1.19	.62966414	1.34	.63976962
.90	.60744854	1.05	.61942537	1.20	.63036427	1.35	.64041454
.91	.60828279	1.06	.62018509	1.21	.63106046	1.36	.64105606
.92	.60911168	1.07	.62094026	1.22	.63175276	1.37	.64169420
.93	.60993527	1.08	.62169093	1.23	.63244119	1.38	.64232900
.94	.61075362	1.09	.62243714	1.24	.63312580	1.39	.64296049
.95	.61156678	1.10	.62317895	1.25	.63380662	1.40	.64358870

TABLE 1 - Continued

Y	Yb				
1.4	.6435887	7.2	.8010527	13.2	.8536588
1.6	.6555077	7.4	.8035935	13.4	.8548549
1.8	.6663340	7.6	.8060539	13.6	.8560277
2.0	.6762335	7.8	.8084382	13.8	.8571778
2.2	.6853382	8.0	.8107500	14.0	.8583059
2.4	.6937540	8.2	.8129931	14.2	.8594127
2.6	.7015677	8.4	.8151707	14.4	.8604989
2.8	.7088511	8.6	.8172861	14.6	.8615651
3.0	.7156639	8.8	.8193420	14.8	.8626118
3.2	.7220568	9.0	.8213412	15.0	.8636397
3.4	.7280727	9.2	.8232862	15.2	.8646493
3.6	.7337487	9.4	.8251795	15.4	.8656411
3.8	.7391165	9.6	.8270232	15.6	.8666156
4.0	.7442040	9.8	.8288196	15.8	.8675733
4.2	.7490353	10.0	.8305705	16.0	.8685146
4.4	.7536319	10.2	.8322778	16.2	.8694401
4.6	.7580126	10.4	.8339433	16.4	.8703402
4.8	.7621942	10.6	.8355687	16.6	.8712452
5.0	.7661916	10.8	.8371555	16.8	.8721255
5.2	.7700182	11.0	.8387052	17.0	.8729916
5.4	.7736861	11.2	.8402193	17.2	.8738438
5.6	.7772060	11.4	.8416989	17.4	.8746825
5.8	.7805879	11.6	.8431455	17.6	.8755080
6.0	.7838406	11.8	.8445602	17.8	.8763207
6.2	.7869722	12.0	.8459441	18.0	.8771208
6.4	.7899901	12.2	.8472983		
6.6	.7929012	12.4	.8486239		
6.8	.7957116	12.6	.8499219		
7.0	.7984270	12.8	.8511931		
		13.0	.8524384		

TABLE 2

GAMMA FUNCTION $\Gamma(n)$

n	0	1	2	3	4	5	6	7	8	9
1.0	1.0000	.9443	.9858	.9835	.9784	.9735	.9687	.9642	.9597	.9555
1.1	.9514	.9474	.9436	.9399	.9364	.9330	.9298	.9267	.9237	.9209
1.2	.9182	.9156	.9131	.9108	.9085	.9064	.9044	.9025	.9007	.8990
1.3	.8975	.8956	.8936	.8914	.8922	.8912	.8902	.8893	.8885	.8879
1.4	.8873	.8868	.8864	.8860	.8858	.8857	.8856	.8856	.8857	.8859
1.5	.8862	.8866	.8870	.8876	.8882	.8889	.8896	.8905	.8914	.8924
1.6	.8935	.8947	.8959	.8972	.8986	.9001	.9017	.9033	.9050	.9068
1.7	.9086	.9106	.9126	.9147	.9168	.9191	.9214	.9238	.9262	.9288
1.8	.9314	.9341	.9368	.9397	.9426	.9456	.9487	.9518	.9551	.9584
1.9	.9618	.9652	.9688	.9724	.9761	.9799	.9837	.9877	.9917	.9958

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx. \quad \Gamma(n) = (n-1) \cdot \Gamma(n-1), \quad n > 1.$$

$\Gamma(n) = (n-1)$ when n is a positive integer.

APPENDIX 1

Computer Program for Fitting a Gamma
Distribution to a Set of Data

This appendix contains the flowchart, the program, and the expected output of a program which fits a gamma distribution to n data points. In addition, the input formats are given on standard forms. The process is as in the main portion of this paper with the following exceptions.

1. In lieu of Table 1, the following equations (5) were used to solve for b .

$$b = y^{-1}(.5000876 + .1648852y - .0544274y^2)$$

For $0 < y < .5772$

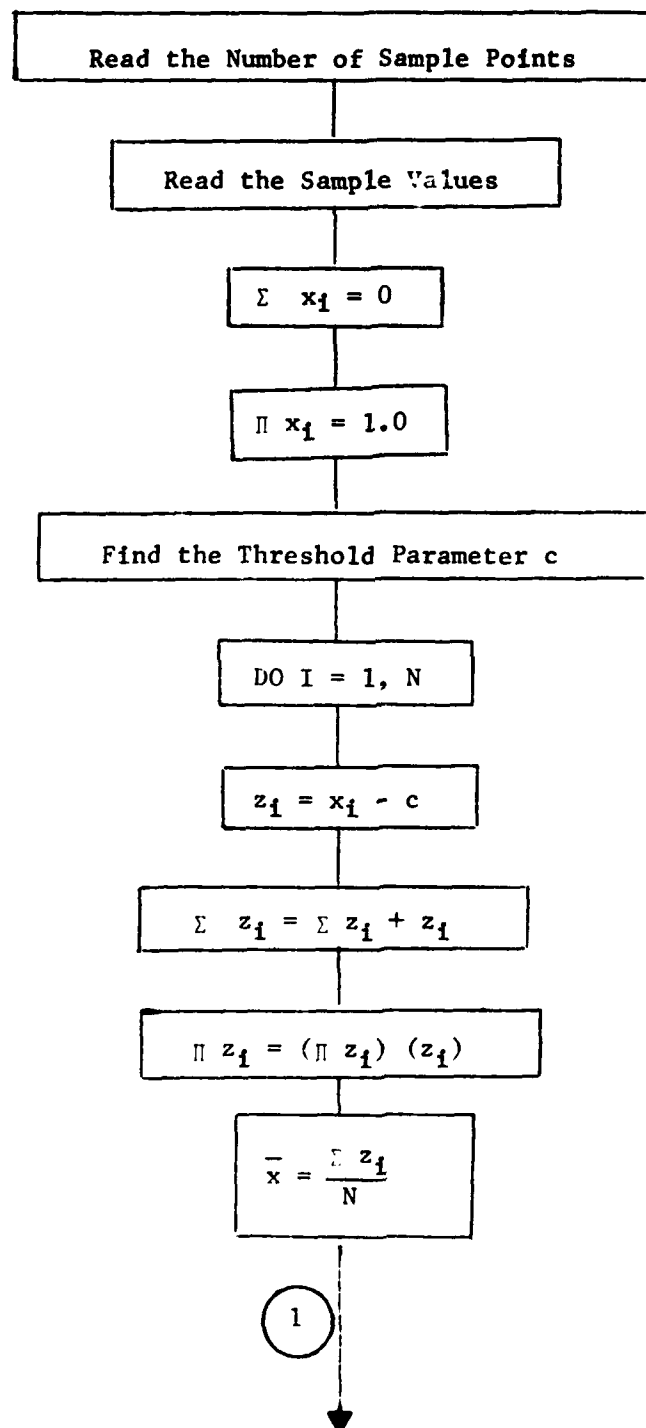
$$b = \frac{8.898919 + 9.059950y + .9775373y^2}{y(17.79728 + 11.968477y + y^2)}$$

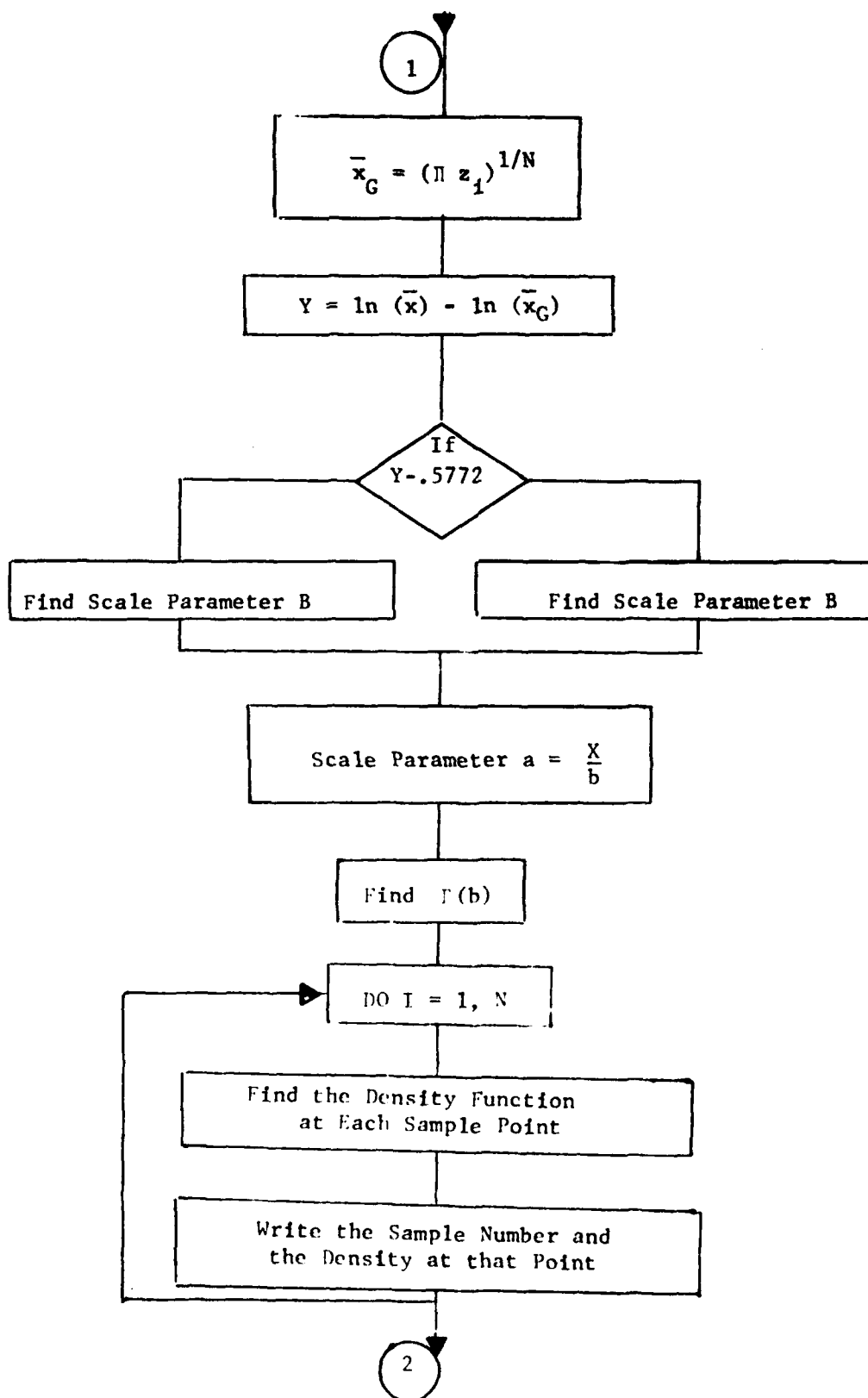
For $.5772 < y < 17.0$

2. The geometric mean was calculated by

$$\bar{t}_G = \frac{1}{n} \sum_{i=1}^n t_i$$

FLOWCHART





2

Write the Values of the
Threshold, Scale and Shape Parameters

```

      DIMENSION X(100),Z(100)
      READ(5,10) N
      READ(5,20) (X(M), M=1,N)
C     THERE ARE N SAMPLE POINTS X(1),X(2),...X(N)
      SXBAR = 0.0
      SXDOT = 1.0
      AN = N
      C=(X(2)+X(1))/2.0-(X(3)-X(1))*(X(3)-X(2))/(4.*X(2)-2.
1 *X(1)-2.*X(3))
C     C IS THE THRESHOLD PARAMETER
      DO 100 L = 1, N
      Z(L) = X(L)-C
      SXBAR = SXBAR + Z(L)
      SXDOT = SXDOT * Z(L)**(1.0/AN)
100  CONTINUE
      XBAR = SXBAR/AN
      Y = ALOG(XBAR/SXDOT)
C     XBAR IS THE ARITHMETIC MEAN, SXDOT THE GEOMETRIC MEAN
      IF(Y-.5772) 1,1,2
1     B = (.5000876+.1648852*Y+.0544274*Y*Y)/Y
      GO TO 3
2     B = (8.898919+9.05995*Y+.9775373*Y*Y)/(Y*(17.79728
1 + 11.968477 * Y + Y*Y))
C     B IS THE SHAPE PARAMETER
3     A = XBAR/B
C     A IS THE SCALE PARAMETER
C     VALUE OF THE DENSITY FUNCTION AT EACH SAMPLE POINT.
C     THE NEXT SIXTEEN CARDS ARE OPTIONAL AND WRITE THE
C     THE FIRST SEVEN CARDS FIND GAMMA(B)
      WRITE(6,30)
30  FORMAT(1H1,14X,'SAMPLE',3X,'SAMPLE',13X,'DENSITY'/
1 14X,'NUMBER',9X,'POINT'//)
      AMUL = 1.0
      BG=B
7     IF(BG - 2.0) 4,4,5
5     BG = BG - 1.0
      AMUL = AMUL * BG
      GO TO 7
4     Q = BG - 1.0
      GAMB=AMUL*(1.-.57710166*Q+.98585399*Q**2.-.87642128*Q
1 **3+.8328212*Q**4.-.5684729*Q**5+.25482049*Q**6.-
2 .0514993*Q**7.)
      DO 200 K = 1,N
      FX = ((Z(K)/A)**(B-1.)*EXP((-1.)*Z(K)/A))/(A*GAMB)
      WRITE(6,40) K, X(K), FX
40  FORMAT(15X,13,10X,F6.2,5X,F16.7)
200  CONTINUE
      WRITE(6,50) C, A, B
10  FORMAT(13)
20  FORMAT(12F6.2)

```



```
50  FORMAT(1H0,'THE DATA IS GAMMA DISTRIBUTED WITH ',  
1'THRESHOLD PARAMETER = ',F7.4//40X,'SCALE PARAMETER = '  
2',F8.4//36X,'AND SHAPE PARAMETER = ',F8.4)  
STOP  
END
```

CHAPTER III

THE GAMMA DISTRIBUTION

In this chapter, the gamma distribution will be investigated. The moments and the moment generating function along with the convolution of the gamma distribution will be given and the method of parameter estimation will be developed.

The gamma density function is given by

$$f(t) = \begin{cases} \frac{\left(\frac{t-c}{a}\right)^{b-1} e^{-\frac{t-c}{a}}}{a \Gamma(b)} & c < t < \infty \\ 0 & \text{otherwise} \end{cases}$$

where c is the threshold value,

a is the scale parameter,

b is the shape parameter, and

$\Gamma(n)$ is the gamma function with argument n (Table 2).

The cumulative distribution is

$$F(t_0) = \Pr(t < t_0) = I(u, p)$$

where $I(u, p)$ is the Incomplete Gamma Function Ratio and in Harter's notation (8)

$$p = b - 1$$

and

$$u = \frac{t_0 - c}{a^{b-1}}$$

When c is not known, it is found to be

SAMPLE NUMBER	SAMPLE POINT	DENSITY
1	0.58	0.0659849
2	0.89	0.2518429
3	0.10	0.3828394
4	1.27	0.4624169
5	1.42	0.5057862
6	1.55	0.5227300
7	1.67	0.5229904
8	1.78	0.5122156
9	1.89	0.4929394
10	2.00	0.4671661
11	2.11	0.4367902
12	2.22	0.4035086
13	2.33	0.3687756
14	2.45	0.3306277
15	2.58	0.2903473
16	2.73	0.2466032
17	2.90	0.2018121
18	3.11	0.1544704
19	3.42	0.1006275

THE DATA IS GAMMA DISTRIBUTED WITH THRESHOLD PARAMETER = 0.1890

SCALE PARAMETER = 0.3886

AND SHAPE PARAMETER = 4.6609

INPUT FORMAT COLUMN IDENTIFICATION

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

NUMBER OF SAMPLE POINTS, N, FORMAT (I3) 0 1 9

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

NUMBER OF SAMPLE POINTS

[illegible][illegible]

FORMAT (12F6.2) 1 . 3 6

SAMPLE VALUES

APPENDIX 2

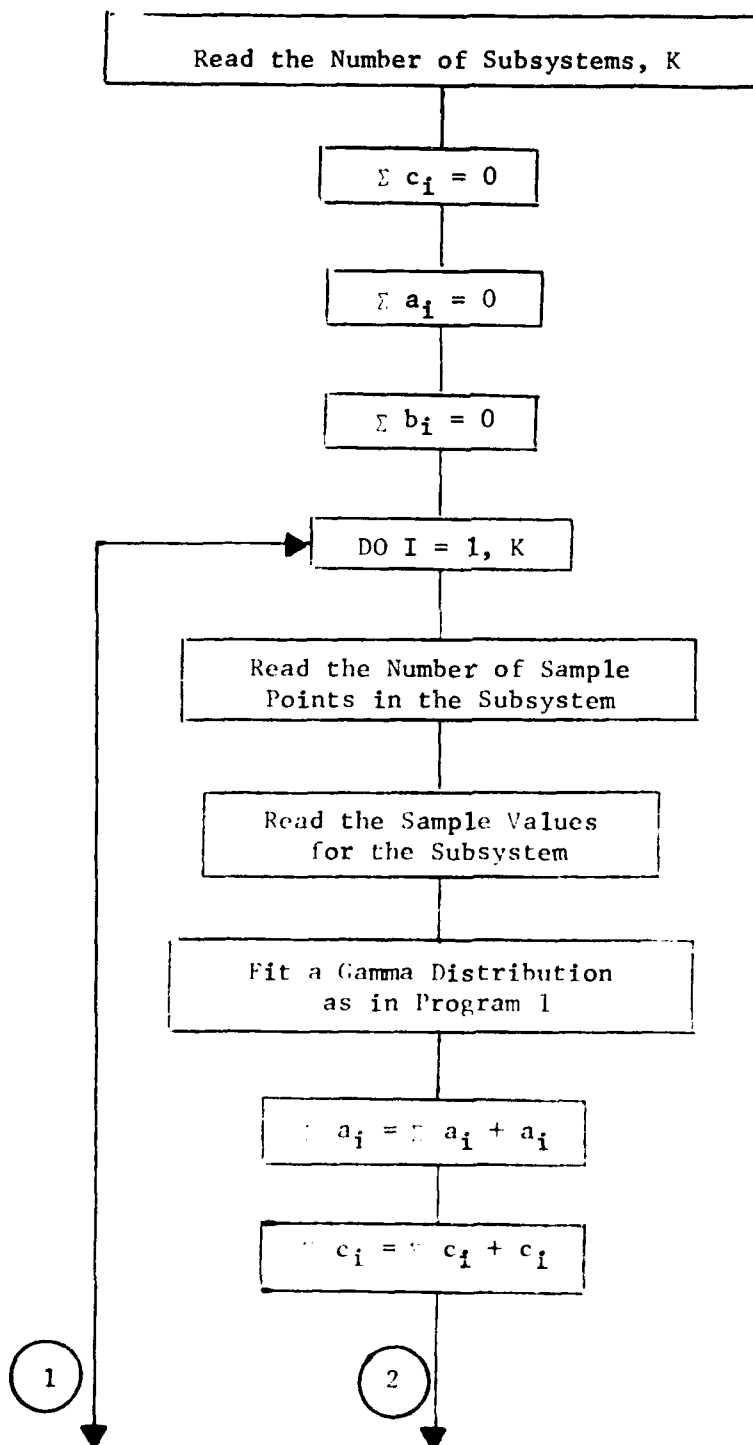
Computer Program for Convolving
Several Sets of Data

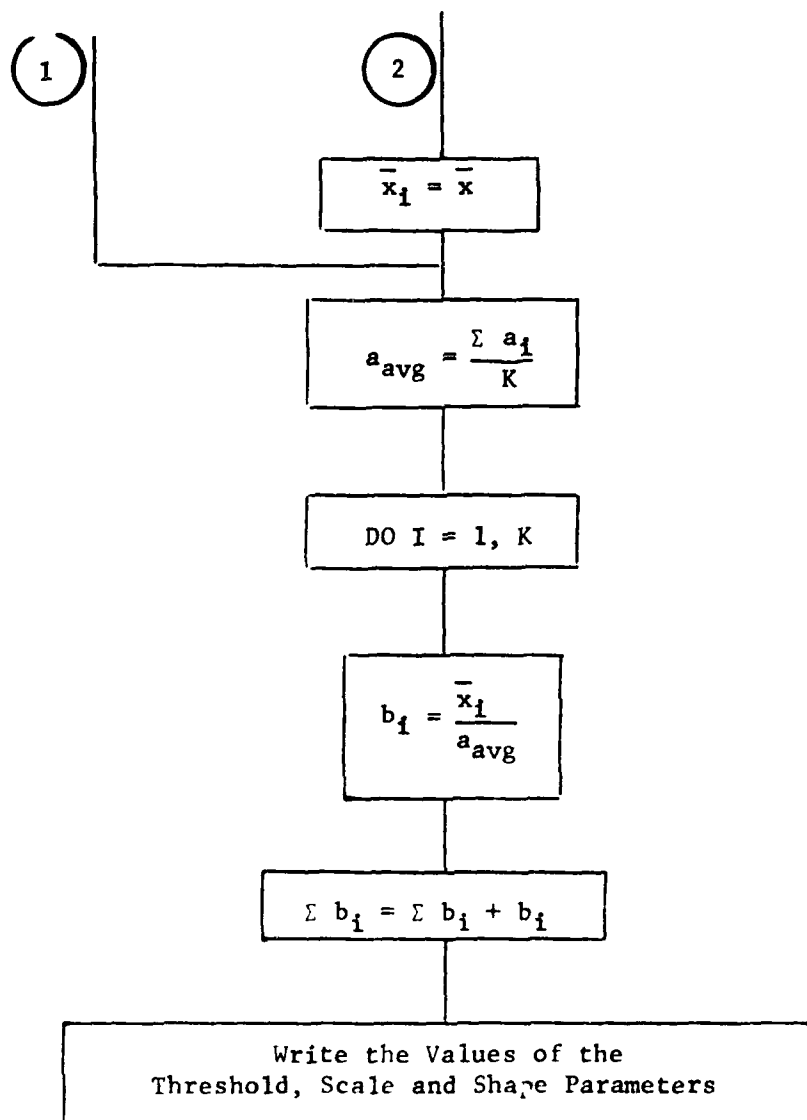
This program utilizes the previous program in a do loop to fit gamma distributions to several sets of data. The average scale parameter is found and each set of data is forced to have this common scale parameter. The distributions are convolved with the threshold

parameter being $\sum_{i=1}^k c_i$, scale parameter = $\frac{\sum_{i=1}^k a_i}{k}$ and shape parameter =

$\frac{\sum_{i=1}^k b_i}{k}$ where there are k sets of data.

FLOWCHART






```

      DIMENSION X(100)
      READ(5,10) K
C     THERE ARE K SUB-SYSTEMS BEING ANALYZED
      SUMC = 0.0
      SUMA = 0.0
      SUMB = 0.0
C     THE DO 800 LOOP FITS A GAMMA DISTRIBUTION TO EACH SET
C     OF DATA.
      DO 800 I = 1,K
      READ(5,10) N
      READ(5,20) (X(M), M=1,N)
C     THERE ARE N SAMPLE POINTS X(1),X(2),...X(N)
      SXBAR = 0.0
      SXDOT = 1.0
      AN = N
      C=(X(2)+X(1))/2.0-(X(3)-X(1))*X(3)-X(2))/(4.*X(2)
1     - 2.*X(1) - 2.*X(3))
C     C IS THE THRESHOLD PARAMETER
      DO 100 L = 1, N
      Z(L) = X(L)-C
      SXBAR = SXBAR + Z(L)
      SXDOT = SXDOT * Z(L)**(1.0/AN)
100    CONTINUE
      XBAR = SXBAR/AN
      XB(I) = XBAR
      Y = ALOG(XBAR/SXDOT)
C     XBAR IS THE ARITHMETIC MEAN, SXDOT THE GEOMETRIC MEAN
      IF(Y-.5772) 1,1,2
1     B = (.5000876+.1648852*Y-.0544274*Y*Y)/Y
      GO TO 3
2     B = (8.898919+9.05995*Y+.9775373*Y*Y)/(Y*(17.79728
1     + 11.968477 * Y + Y*Y))
C     B IS THE SHAPE PARAMETER
3     A = XBAR/B
C     A IS THE SCALE PARAMETER
      SUMA = SUMA + A
      SUMC = SUMC + C
C     THIS WRITE STATEMENT IS OPTIONAL AND PRINTS THE
C     PARAMETERS OF EACH SUB-SYSTEM.
      WRITE(6,30) I, C, A, B
30    FORMAT(1H0,'SUB-SYSTEM',I3,'HAS BEEN FITTED WITH A',
1     ' GAMMA DISTRIBUTION AND'//' C ='F8.4,' , A ='F8.4,
2     ' AND B = ',F8.4)
800    CONTINUE
      AK = K
      AAVG = SUMA/AK
      DO 900 L = 1,K
      SUMB = SUMB + XB(L)/AAVG
900    CONTINUE
      WRITE(6,40) K, SUMC, AAVG, SUMB
10    FORMAT(I3)

```

```
20  FORMAT(12F6.2)
40  FORMAT(1H0,I3,' SUBSYSTEMS HAVE BEEN FITTED WITH GAMMA'
    1,' DISTRIBUTIONS AND THEIR CONVOLUTION IS'/' A GAMMA'
    2,' DISTRIBUTION WITH C = ',F8.4,' A =',F8.4,' AND B ='
    3F8.4)
    STOP
```

SUB-SYSTEM 1 HAS BEEN FITTED WITH A GAMMA DISTRIBUTION AND
 $C = 0.9400$, $A = 0.4600$ AND $B = 4.4778$

SUB-SYSTEM 2 HAS BEEN FITTED WITH A GAMMA DISTRIBUTION AND
 $C = 2.6793$, $A = 0.5364$ AND $B = 4.3263$

SUB-SYSTEM 3 HAS BEEN FITTED WITH A GAMMA DISTRIBUTION AND
 $C = 2.0327$, $A = 0.03400$ AND $B = 11.6674$

SUB-SYSTEM 4 HAS BEEN FITTED WITH A GAMMA DISTRIBUTION AND
 $C = 0.1890$, $A = 0.3886$ AND $B = 4.6609$

4 SUBSYSTEMS HAVE BEEN FITTED WITH GAMMA DISTRIBUTIONS AND THEIR CONVOLUTION IS
A GAMMA DISTRIBUTION WITH $C = 5.8410$ $A = 0.4313$ AND $B = 23.5563$

INPUT FORMAT COLUMN IDENTIFICATION

NUMBER OF SUBSYSTEMS, K, FORMAT (I3) 0 0 4

NUMBER OF SUBSYSTEMS

NUMBER OF SAMPLE POINTS

INPUT FORMAT COLUMN IDENTIFICATION

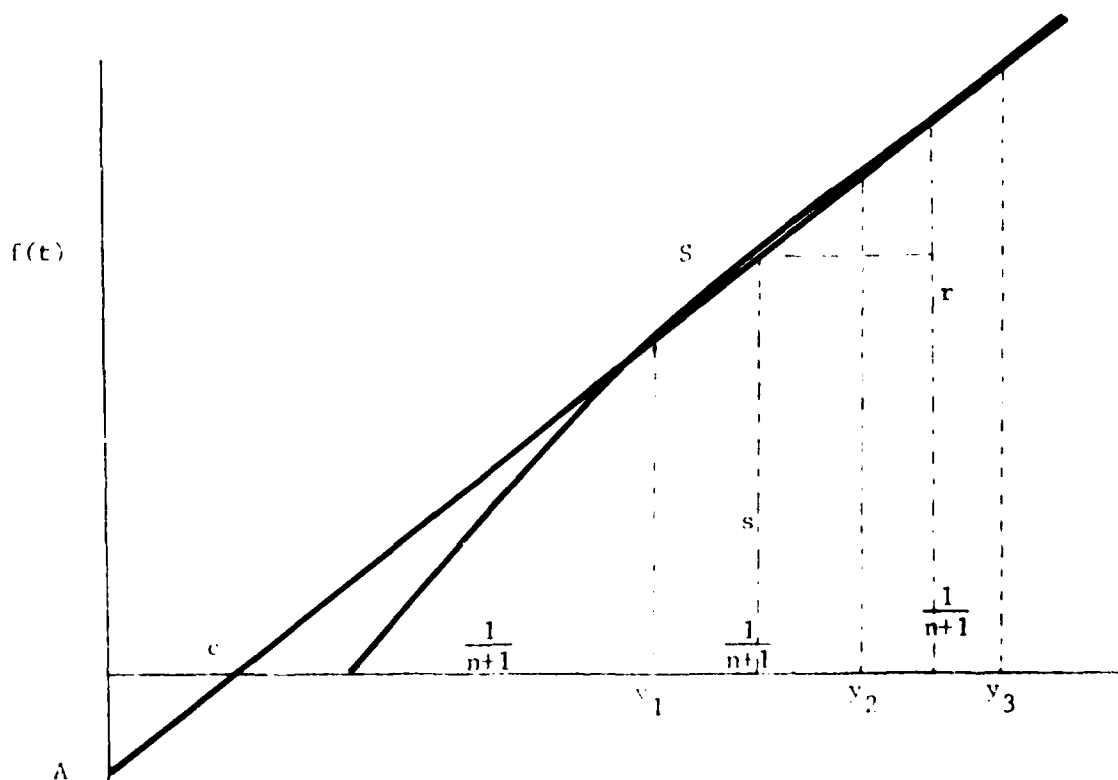
SAMPLE VALUES, X (I)
 FORMAT (12F6.2) -- 1 3 6

SAMPLE VALUES

APPENDIX 3

Derivation of the Equation
for the Threshold Parameter

As indicated in Chapter III, the maximum likelihood equation for the threshold parameter requires an iteration process. As this parameter is rarely known in maintainability analysis, some estimate of this parameter must be made from the data available. Since the area between adjacent ordered sample points is known from order statistics to be $\frac{1}{n+1}$ (10), a straight line can be fitted to the unknown curve and the threshold parameter estimated.



The area between the first and second sample points is

$$\frac{1}{n+1} = s(y_2 - y_1)$$

and the area between the second and third points is

$$\frac{1}{n+1} = r(y_3 - y_2).$$

The slope of the line is then

$$\text{slope} = \frac{2(r - s)}{y_3 - y_1}$$

and at the point S

$$s = A + \text{slope} \frac{y_2 + y_1}{2}$$

$$A = s - \text{slope} \frac{y_2 + y_1}{2} \quad \text{Equation 4}$$

At the point c

$$0 = A + \text{slope} (c)$$

$$A = - \text{slope} (c). \quad \text{Equation 5}$$

Equating equation 4 to equation 5

$$s - \text{slope} \frac{y_2 + y_1}{2} = - \text{slope} (c)$$

and

$$c = \frac{y_2 + y_1}{2} = - \frac{s}{\text{slope}}$$

$$= \frac{y_2 + y_1}{2} - \frac{(y_3 - y_2)(y_3 - y_1)}{2(2y_2 - y_1 - y_3)}$$

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CHAPTER III

THE GAMMA DISTRIBUTION

In this chapter, the gamma distribution will be investigated. The moments and the moment generating function along with the convolution of the gamma distribution will be given and the method of parameter estimation will be developed.

The gamma density function is given by

$$f(t) = \begin{cases} \frac{\left(\frac{t-c}{a}\right)^{b-1} e^{-\frac{t-c}{a}}}{a \Gamma(b)} & c < t < \infty \\ 0 & \text{otherwise} \end{cases}$$

where c is the threshold value,

a is the scale parameter,

b is the shape parameter, and

$\Gamma(n)$ is the gamma function with argument n (Table 2).

The cumulative distribution is

$$F(t_0) = \Pr(t < t_0) = I(u, p)$$

where $I(u, p)$ is the Incomplete Gamma Function Ratio and in Harter's notation (8)

$$p = b - 1$$

and

$$u = \frac{t_0 - c}{a}$$

The mean and variance are found to be